Math 155, *Lecture Notes- Bonds* **Name**____________

***Section 8.5** Partial Fractions* → $\boxed{\int \frac{1}{u}du = \ln|u| + C}$ ← Will be a Key formula.

In this section we will consider the *method of partial fractions*, a technique for decomposing a rational function into simpler rational functions.

When the numerator is not the derivative of the denominator, nor is it a constant multiple of the derivative of the denominator, a simple substitution won't work. However, if we notice that the integrand can be decomposed, then the integral is actually quite simple:

$$\int \frac{x}{x^2-5x+6}dx = \int \left[\frac{-2}{x-2} + \frac{3}{x-3}\right]dx$$

$$= -2\int \frac{1}{x-2}dx + 3\int \frac{1}{x-3}dx = -2\ln|x-2| + 3\ln|x-3| + C$$

Ex.1 Decompose: $\frac{x}{x^2-5x+6}$

Let $\frac{x}{x^2-5x+6} = \frac{A}{x-2} + \frac{B}{x-3}$, where A & B are constants.

$(x-2)(x-3)\cdot\left[\frac{x}{x^2-5x+6}\right] = (x-2)(x-3)\cdot\left[\frac{A}{x-2} + \frac{B}{x-3}\right]$ ← clears the denominators

$x = A\cdot(x-3) + B(x-2)$ ← can we find A & B? (BASIC EQUATION)

$x = Ax - 3A + Bx - 2B$

$1x + 0 = (Ax + Bx) + (-3A - 2B)$ Match coefficients

$1\cdot x + 0 = (A+B)x + (-3A-2B)$ to create a system of equations

$\begin{cases} A + B = 1 \\ -3A - 2B = 0 \end{cases}$ ⟹ Matrix: $\left[\begin{array}{cc|c} 1 & 1 & 1 \\ -3 & -2 & 0 \end{array}\right]$ ← Use "rref" to solve the system on Ti-83, or Ti-84.

Solve by Addition/Elimination:

First → $3(A+B) = 3(1)$ ← "Prep" for opposite coefficients

$3A + 3B = 3$

$+\ -3A - 2B = 0$

$B = 3$ ⟹, Find A! $A + B = 1$

$A + (3) = 1$

$A = -2$

So,

$\frac{x}{x^2-5x+6} = \frac{-2}{x-2} + \frac{3}{x-3}$

Ex.2 Integrate: $\int \frac{3x+11}{x^2-x-6}dx = \int \left[\frac{4}{x-3} + \frac{-1}{x+2}\right]dx$

$= 4\int \frac{1}{x-3}dx - \int \frac{1}{x+2}dx$

$= 4\int \frac{1}{u}du - \int \frac{1}{w}dw$

$= 4\cdot[\ln|u|] - [\ln|w|] + C$

$\boxed{= 4\ln|x-3| - \ln|x+2| + C}$

check!??

Let $u = x-3$

$\frac{du}{dx} = 1$

$du = dx$

Let $w = x+2$

$\frac{dw}{dx} = 1$

$dw = dx$

– Partial Fractions:

Let $\frac{3x+11}{x^2-x-6} = \frac{A}{x-3} + \frac{B}{x+2}$, where A & B are constants

$(x-3)(x+2)\cdot\left[\frac{3x+11}{x^2-x-6}\right] = (x-3)(x+2)\cdot\left[\frac{A}{x-3} + \frac{B}{x+2}\right]$

$3x+11 = A(x+2) + B(x-3)$ ← Can we find A & B? (BASIC EQUATION)

$3x+11 = Ax+2A+Bx-3B$

$3x+11 = (A+B)x + (2A-3B)$

Match coefficients to create a system of equations

$\begin{cases} A+B=3 \\ 2A-3B=11 \end{cases} \Rightarrow$ Matrix: $\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & -3 & 11 \end{array}\right]$ ← Use "rref" to solve on Ti-83 or Ti-84.

Solve by Addition/Elimination:

first → $3(A+B) = 3\cdot 3$ ← "Prep" for opposite coefficients

$3A+3B = 9$

$+\ 2A-3B = 11$

$5A = 20$

$\frac{5A}{5} = \frac{20}{5}$

$A = 4$ ⇒ Find B: $A+B=3$

$(4)+B=3$

$B=-1$

So,

$\boxed{\frac{3x+11}{x^2-x-6} = \frac{4}{x-3} + \frac{-1}{x+2}}$

Ex.3 Integrate: $\int \frac{1}{4x^2-9}dx = ??$ Use Partial Fractions!

Decompose: $\frac{1}{4x^2-9} = \frac{1}{(2x+3)(2x-3)}$.

Let $\frac{1}{4x^2-9} = \frac{A}{2x+3} + \frac{B}{2x-3}$, where A & B are constants.

$(2x+3)(2x-3)\cdot\left[\frac{1}{4x^2-9}\right] = (2x+3)(2x-3)\cdot\left[\frac{A}{2x+3} + \frac{B}{2x-3}\right]$

$1 = A(2x-3) + B\cdot(2x+3)$ ← Can we find A & B? (BASIC EQUATION)

$1 = 2Ax - 3A + 2Bx + 3B$ Match coefficients to create a system of equations

$0\cdot x + 1 = (2A+2B)x + (-3A+3B)$

$\begin{cases} 2A+2B=0 \\ -3A+3B=1 \end{cases} \Longrightarrow$ Matrix: $\left[\begin{array}{cc|c} 2 & 2 & 0 \\ -3 & 3 & 1 \end{array}\right]$ ← Use "rref" to solve the system on Ti-83, or Ti-84.

First: $3(2A+2B) = 3\cdot 0$ "Prep" for opposite coefficients
& $2\cdot(-3A+3B) = 2\cdot 1$

$$\begin{aligned} 6A+6B&=0 \\ +\ -6A+6B&=2 \\ \hline 12B&=2 \end{aligned}$$

$\frac{12B}{12} = \frac{2}{12}$

$B = \frac{1}{6} \Rightarrow$ Find A: $2A+2B=0$

$2A + 2(\frac{1}{6}) = 0$

$2A + \frac{1}{3} = 0$

$2A = -\frac{1}{3}$

$\frac{1}{2}\cdot\frac{2A}{1} = -\frac{1}{3}\cdot\frac{1}{2}$

$A = -\frac{1}{6}$

So, $\boxed{\frac{1}{4x^2-9} = \frac{-\frac{1}{6}}{2x+3} + \frac{\frac{1}{6}}{2x-3}}$

Ex.3 cont'd

$$\int \frac{1}{4x^2-9}dx = \int \left[\frac{-\frac{1}{6}}{2x+3} + \frac{\frac{1}{6}}{2x-3}\right]dx$$

$$= -\frac{1}{6}\cdot\int \frac{1}{2x+3}dx + \frac{1}{6}\int\frac{1}{2x-3}dx$$

$$= -\frac{1}{6}\int \frac{1}{u}\left(\frac{du}{2}\right) + \frac{1}{6}\int\frac{1}{w}\left(\frac{dw}{2}\right)$$

$$= -\frac{1}{12}\cdot\int \frac{1}{u}du + \frac{1}{12}\int\frac{1}{w}dw$$

$$= -\frac{1}{12}\cdot[\ln|u|] + \frac{1}{12}\cdot[\ln|w|] + C$$

$$= -\frac{1}{12}\cdot\ln|2x+3| + \frac{1}{12}\cdot\ln|2x-3| + C$$

$$= \frac{1}{12}\ln|2x-3| - \frac{1}{12}\ln|2x+3| + C$$

$$= \frac{1}{12}\cdot\left[\ln|2x-3| - \ln|2x+3|\right] + C$$

$$\boxed{= \frac{1}{12}\ln\left|\frac{2x-3}{2x+3}\right| + C}$$

check: ??

Let $u = 2x+3$

$\frac{du}{dx} = 2$

$\frac{du}{2} = dx$

Let $w = 2x-3$

$\frac{dw}{dx} = 2$

$\frac{dw}{2} = dx$

Ex.4 Evaluate: $\int \frac{4x^2}{x^3+x^2-x-1}dx = ??$ Use Partial Fractions !

Decompose: $\frac{4x^2}{x^3+x^2-x-1} = \frac{4x^2}{x^2(x+1)-(x+1)} = \frac{4x^2}{(x+1)(x^2-1)} = \frac{4x^2}{(x+1)(x+1)(x-1)}$

$$\frac{4x^2}{x^3+x^2-x-1} = \frac{4x^2}{(x-1)(x+1)^2}$$

Let $\frac{4x^2}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$, where A, B, & C are constants.

$$(x-1)(x+1)^2 \cdot \left[\frac{4x^2}{(x-1)(x+1)^2}\right] = (x-1)(x+1)^2 \cdot \left[\frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}\right]$$

$4x^2 = A(x+1)^2 + B(x-1)(x+1) + C(x-1)$ ← can we find A, B & C ?

$$4x^2+0x+0 = A(x^2+2x+1) + B(x^2-1) + Cx - C$$
$$4x^2+0x+0 = Ax^2+2Ax+A+Bx^2-B+Cx-C$$
$$4x^2+0x+0 = (A+B)x^2 + (2A+C)x + (A-B-C)$$

Match coefficients to create a system of Equations

$$\begin{cases} A+B=4 \\ 2A+C=0 \\ A-B-C=0 \end{cases} \Longrightarrow \begin{cases} A+B+0C=4 \\ 2A+0B+C=0 \\ A-B-C=0 \end{cases} \Rightarrow$$

MATRIX:
$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 4 \\ 2 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 \end{array}\right]$$

Use "rref" to solve the system on Ti-83, or Ti-84.

$$\begin{array}{r} 2A+C=0 \\ +\ A-B-C=0 \\ \hline 3A-B=0 \end{array} \qquad \begin{array}{r} 3A-B=0 \\ +\ A+B=4 \\ \hline 4A=4 \end{array}$$

$$\frac{4A}{4}=\frac{4}{4}$$
$$A=1$$

Find B: $A+B=4$
$(1)+B=4$
$B=3$

Find C:
$2A+C=0$
$2(1)+C=0$
$2+C=0$
$C=-2$

$$\boxed{\text{So, } \frac{4x^2}{x^3+x^2-x-1} = \frac{1}{x-1} + \frac{3}{x+1} + \frac{-2}{(x+1)^2}}$$

Ex. 4 cont'd

$$\int \frac{\cancel{4x}\,4x^2}{x^3+x^2-x-1}dx = \int \left[\frac{1}{x-1} + \frac{3}{x+1} + \frac{-2}{(x+1)^2}\right]dx$$

Let $u = x-1$

$\frac{du}{dx} = 1$

$du = dx$

Let $w = x+1$

$\frac{dw}{dx} = 1$

$dw = dx$

$$= \int \frac{1}{x-1}dx + 3\int \frac{1}{x+1}dx - 2\int \frac{1}{(x+1)^2}dx$$

$$= \int \frac{1}{u}du + 3\int \frac{1}{w}dw - 2\int w^{-2}dw$$

$$= \ln|u| + 3\cdot[\ln|w|] - 2\cdot[-1\cdot w^{-1}] + C$$

$$= \ln|x-1| + 3\ln|x+1| + \frac{2}{w} + C$$

$$\boxed{= \ln|x-1| + 3\ln|x+1| + \frac{2}{x+1} + C}$$

check??

Let's learn an interesting method of solving the BASIC EQUATIONS. I call it the "method of choice."

From Example 1, consider $x = A\cdot(x-3) + B\cdot(x-2)$.

To solve for A & B, choose $x = 3$, and we get the following:

$$(3) = A\cdot[(3)-3] + B\cdot[(3)-2]$$

$$3 = A\cdot 0 + B\cdot 1$$

$$3 = B$$

Now, choose $x = 2$, and we get the following:

$$(2) = A\cdot[(2)-3] + 3[(2)-2]$$

$$2 = A\cdot[-1] + 3\cdot 0$$

$$2 = -A$$

$$A = -2$$

What do you think?

Let's revisit Examples 2, 3 & 4.

From Example 2, consider $3x+11 = A\cdot(x+2) + B(x-3)$.
To solve for A & B, choose $x = 3$, and we get the following:

$$3(3)+11 = A[(3)+2] + B\cdot[(3)-3]$$
$$9+11 = A\cdot[5] + \underline{B\cdot[0]}$$
$$20 = 5A$$
$$\frac{20}{5} = \frac{5A}{5}$$
$$\underline{4 = A}$$

Now, choose $x = -2$, and we get the following:

$$3(-2)+11 = 4\cdot[(-2)+2] + B\cdot[(-2)-3]$$
$$-6+11 = \underline{4\cdot[0]} + B\cdot[-5]$$
$$5 = -5B$$
$$\frac{5}{-5} = \frac{-5B}{-5}$$
$$\underline{-1 = B}$$

From Example 3, consider $1 = A\cdot(2x-3) + B\cdot(2x+3)$.
To solve for A & B, choose $x = \frac{3}{2}$, and we get the following:

$$1 = A\cdot[2(\tfrac{3}{2})-3] + B\cdot[2(\tfrac{3}{2})+3]$$
$$1 = A\cdot[3-3] + B\cdot[3+3]$$
$$1 = \underline{A\cdot[0]} + 6B$$
$$1 = 6B$$
$$\underline{B = \tfrac{1}{6}}$$

Now, choose $x = -\frac{3}{2}$, and we get the following:

$$1 = A\cdot[2(\tfrac{-3}{2})-3] + \tfrac{1}{6}\cdot[2(\tfrac{-3}{2})+3]$$
$$1 = A\cdot[-3-3] + \tfrac{1}{6}\cdot[-3+3]$$
$$1 = A\cdot[-6] + \underline{\tfrac{1}{6}\cdot[0]}$$
$$1 = -6A$$
$$\underline{A = -\tfrac{1}{6}}$$

From Example 4, consider $4x^2 = A(x+1)^2 + B(x-1)(x+1) + C(x-1)$.

To solve for A, B, & C, choose $x = 1$, and we get the following:

$4(1)^2 = A\cdot[(1)+1]^2 + B[(1)-1]\cdot[(1)+1] + C\cdot[(1)-1]$

$4 = A\cdot[2]^2 + \underline{\underline{B\cdot[0]\cdot[2]}} + \underline{\underline{C\cdot[0]}}$

$4 = A\cdot 4$

$\underline{A = 1}$

Now, choose $x = -1$, and we get the following:

$4(-1)^2 = 1\cdot[(-1)+1]^2 + B\cdot[(-1)-1][(-1)+1] + C[(-1)-1]$

$4\cdot 1 = \underline{\underline{1\cdot[0]^2}} + \underline{\underline{B\cdot[-2]\cdot[0]}} + C\cdot[-2]$

$4 = -2C$

$\frac{4}{-2} = \frac{-2C}{-2}$

$\underline{-2 = C}$

Now, choose $x = 0$ and we get the following:

$4(0)^2 = 1\cdot[(0)+1]^2 + B\cdot[(0)-1]\cdot[(0)+1] + (-2)[(0)-1]$

$4\cdot 0 = 1\cdot[1]^2 + B\cdot[-1]\cdot[1] + (-2)\cdot[-1]$

$0 = 1 - B + 2$

$0 = 3 - B$

$\underline{B = 3}$

Cool technique! Just make thoughtful, convenient choices and use <u>zero</u> to help get rid of unwanted coefficients.

Decomposition of $N(x)/D(x)$ into Partial Fractions

1. **Divide if improper:** If $N(x)/D(x)$ is an improper fraction (that is, if the degree of the numerator is greater than or equal to the degree of the denominator), divide the denominator into the numerator to obtain

$$\frac{N(x)}{D(x)} = (\text{a polynomial}) + \frac{N_1(x)}{D(x)}$$

where the degree of $N_1(x)$ is less than the degree of $D(x)$. Then apply Steps 2, 3, and 4 to the proper rational expression $N_1(x)/D(x)$.

2. **Factor denominator:** Completely factor the denominator into factors of the form

$$(px + q)^m \qquad \text{and} \qquad (ax^2 + bx + c)^n$$

where $ax^2 + bx + c$ is irreducible.

3. **Linear factors:** For each factor of the form $(px + q)^m$, the partial fraction decomposition must include the following sum of m fractions.

$$\frac{A_1}{(px + q)} + \frac{A_2}{(px + q)^2} + \cdots + \frac{A_m}{(px + q)^m}$$

4. **Quadratic factors:** For each factor of the form $(ax^2 + bx + c)^n$, the partial fraction decomposition must include the following sum of n fractions.

$$\frac{B_1x + C_1}{ax^2 + bx + c} + \frac{B_2x + C_2}{(ax^2 + bx + c)^2} + \cdots + \frac{B_nx + C_n}{(ax^2 + bx + c)^n}$$

Ex.5 FACTOR:

$$\begin{aligned}
&x^5 + x^4 - x - 1 \\
&= x^4(x + 1) - (x + 1) \\
&= (x^4 - 1)(x + 1) \\
&= (x^2 + 1)(x^2 - 1)(x + 1) \\
&= (x^2 + 1)(x + 1)(x - 1)(x + 1) \\
&= (x - 1)(x + 1)^2(x^2 + 1)
\end{aligned}$$

Terminology: $(x - 1)$ is called a <u>linear factor</u>,

$(x + 1)^2$ is called a <u>repeated linear factor</u>, and

$(x^2 + 1)$ is called an <u>irreducible quadratic factor</u>.

Degree 3

"Improper"

Ex.6 Use Long Division, then Integrate: $\int \frac{x^3 - x + 3}{x^2 + x - 2}dx$

Degree 2

1st: LONG DIVISION

$$\begin{array}{r} x - 1 \\ x^2+x-2 \overline{\big)\, x^3 + 0\cdot x^2 - x + 3} \\ -x^3 - x^2 + 2x \\ \hline -x^2 + x + 3 \\ +x^2 + x - 2 \\ \hline 2x + 1 \end{array}$$

← Quotient

← Remainder

$x\cdot(x^2+x-2) = x^3+x^2-2x$

change signs in subtraction

$-1\cdot(x^2+x-2) = -x^2-x+2$

So, $\frac{x^3-x+3}{x^2+x-2} = x - 1 + \frac{2x+1}{x^2+x-2}$ ← Use Partial Fractions Method

$$\frac{2x+1}{x^2+x-2} = \frac{2x+1}{(x+2)(x-1)}$$

$$(x+2)(x-1)\cdot\left[\frac{2x+1}{x^2+x-2}\right] = (x+2)(x-1)\cdot\left[\frac{A}{x+2} + \frac{B}{x-1}\right]$$

$$2x+1 = A\cdot(x-1) + B\cdot(x+2)$$ (Basic Equation)

choose: $x=1$

$2(1)+1 = A\cdot[(1)-1] + B\cdot[(1)+2]$

$2+1 = A\cdot[0] + B\cdot[3]$

$3 = 3B$

$B=1$,

choose: $x=-2$

$2(-2)+1 = A\cdot[(-2)-1] + 1\cdot[(-2)+2]$

$-4+1 = A\cdot[-3] + 1\cdot[0]$

$-3 = -3A$

$A = 1$.

So, we have $\boxed{\frac{2x+1}{x^2+x-2} = \frac{1}{x+2} + \frac{1}{x-1}}$

Ex.6 continued

This means $\dfrac{x^3-x+3}{x^2+x-2} = x-1+\dfrac{1}{x+2}+\dfrac{1}{x-1}$.

$$\int \frac{x^3-x+3}{x^2+x-2}\,dx = \int \left[x-1+\frac{1}{x+2}+\frac{1}{x-1}\right]dx$$

$$= \int x\,dx - \int 1\,dx + \int \frac{1}{x+2}\,dx + \int \frac{1}{x-1}\,dx$$

$$= \frac{x^2}{2} - x + \int \frac{1}{u}\,du + \int \frac{1}{w}\,dw$$

$$= \frac{x^2}{2} - x + \ln|u| + \ln|w| + C$$

$$= \frac{x^2}{2} - x + \ln|x+2| + \ln|x-1| + C$$

$$= \frac{x^2}{2} - x + \ln|(x+2)(x-1)| + C$$

$$\boxed{= \frac{x^2}{2} - x + \ln|x^2+x-2| + C}$$

Let $u = x+2$

$\dfrac{du}{dx} = 1$

$du = dx$

Let $w = x-1$

$\dfrac{dw}{dx} = 1$

$dw = dx$

check: ??

Ex.7 Evaluate: $\int \frac{2x^3-4x-8}{(x^2-x)(x^2+4)}dx$

All Fractions Need to be "Proper"

$$\frac{2x^3-4x-8}{(x^2-x)(x^2+4)} = \frac{2x^3-4x-8}{x(x-1)(x^2+4)}$$

$$x(x-1)(x^2+4)\cdot\left[\frac{2x^3-4x-8}{x\cdot(x-1)(x^2+4)}\right] = x\cdot(x-1)(x^2+4)\cdot\left[\frac{A}{x}+\frac{B}{x-1}+\frac{Cx+D}{x^2+4}\right]$$

Degree 0   Degree 1   Degree 2   Degree 1

Irreducible Quadratic Factor

$$\underline{\underline{2x^3-4x-8 = A\cdot(x-1)(x^2+4) + B(x)(x^2+4) + (Cx+D)(x)(x-1)}}$$ (Basic Equation)

$$2x^3-4x-8 = A\cdot(x^3-x^2+4x-4) + Bx^3+4Bx + (Cx+D)(x^2-x)$$

$$2x^3-4x-8 = Ax^3-Ax^2+4Ax-4A+Bx^3+4Bx+Cx^3-Cx^2+Dx^2-Dx$$

$\boxed{\text{Match Coefficients}}$

$$\begin{aligned}2x^3+0x^2-4x-8 = &(Ax^3+Bx^3+Cx^3)\\ &+(-Ax^2-Cx^2+Dx^2)\\ &+(4Ax+4Bx-Dx)\\ &+(-4A)\end{aligned}$$

<u>System of Equations</u>

$$\begin{cases} A+B+C+0D = 2 \\ -A+0B-C+D = 0 \\ 4A+4B+0C-D = -4 \\ -4A+0B+0C+0D = -8 \end{cases} \Rightarrow \text{MATRIX} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 2 \\ -1 & 0 & -1 & 1 & 0 \\ 4 & 4 & 0 & -1 & -4 \\ -4 & 0 & 0 & 0 & -8 \end{array}\right] = [E]$$

My choice for the NAME

solve on Ti-83 or Ti-84.

rref([E]) gives us

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right] \quad \begin{aligned} A&=2\\ B&=-2\\ C&=2\\ D&=4\end{aligned}$$

I. MATRX menu
II. EDIT (To NAME Your MATRIX
III. 4 Rows X 5 columns (size)
IV. Place numerical entries
V. MATRX MENU
VI. MATH MENU
VII. rref([E])

Ex.7 continued So, we have $\boxed{\dfrac{2x^3-4x-8}{(x^2-x)(x^2+4)} = \dfrac{2}{x} + \dfrac{-2}{x-1} + \dfrac{2x+4}{x^2+4}}$.

$$\int \frac{2x^3-4x-8}{(x^2-x)(x^2+4)}\,dx = \int \left[\frac{2}{x} - \frac{2}{x-1} + \frac{2x+4}{x^2+4}\right] dx$$

$$= 2\int \frac{1}{x}\,dx - 2\int \frac{1}{x-1}\,dx + \int \frac{2x}{x^2+4}\,dx + 4\int \frac{1}{x^2+4}\,dx$$

$$= 2\cdot \ln|x| - 2\int \frac{1}{u}\,du + \int \frac{2x}{w}\left(\frac{dw}{2x}\right) + 4\int \frac{1}{x^2+2^2}\,dx \quad ★$$

$$= 2\ln|x| - 2\cdot\ln|u| + \int \frac{1}{w}\,dw + 4\cdot\left[\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right] + C$$

$$= 2\ln|x| - 2\ln|x-1| + \ln|w| + 2\tan^{-1}\left(\frac{x}{2}\right) + C \leftarrow \text{O.K.}$$

Non-Negative

$$= \ln|x^2| - \ln|(x-1)^2| + \ln|x^2+4| + 2\tan^{-1}\left(\frac{x}{2}\right) + C$$

$$= \ln\left(\frac{x^2}{(x-1)^2}\right) + \ln(x^2+4) + 2\tan^{-1}\left(\frac{x}{2}\right) + C$$

$$\boxed{= \ln\left(\frac{x^2(x^2+4)}{(x-1)^2}\right) + 2\tan^{-1}\left(\frac{x}{2}\right) + C}$$

simplify with LOG RULES - Nice!

Let $u = x-1$

$\frac{du}{dx} = 1$

$du = dx$

Let $w = x^2+4$

$\frac{dw}{dx} = 2x$

$\frac{dw}{2x} = dx$

Check: ??

★ $\displaystyle\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

use $a^2 = 4$

$a = 2$

All Factions need to be "proper"

Ex.8 Evaluate: $\displaystyle\int \frac{x^2+x+3}{x^4+6x^2+9}\,dx$

$$\frac{x^2+x+3}{x^4+6x^2+9} = \frac{x^2+x+3}{(x^2+3)(x^2+3)} = \frac{x^2+x+3}{(x^2+3)^2}$$

↑ Repeat Quadratic Factors

Degree 1

$$(x^2+3)^2\cdot\left[\frac{x^2+x+3}{(x^2+3)^2}\right] = (x^2+3)^2\cdot\left[\frac{Ax+B}{x^2+3} + \frac{Cx+D}{(x^2+3)^2}\right]$$

Degree 2   Degree 2, Repeated

$$\underline{\underline{x^2+x+3 = (Ax+B)(x^2+3) + (Cx+D)}}$$

(BASIC EQUATION)

$$x^2+x+3 = Ax^3+Bx^2+3Ax+3B+Cx+D$$

$$0\cdot x^3+1\cdot x^2+1\cdot x+3 = Ax^3+Bx^2+(3A+C)x+(3B+D)$$

$\boxed{\text{Match Coefficients}}$

System of Equations

$$\begin{cases} A=0 \\ B=1 \\ 3A+C=1 \\ 3B+D=3 \end{cases}$$

Solve using substitution.

Find C: $3A+C=1$

$3(0)+C=1$

$\underline{C=1}$

Find D: $3B+D=3$

$3(1)+D=3$

$3+D=3$

$\underline{D=0}$

So, we have $\boxed{\dfrac{x^2+x+3}{x^4+6x^2+9} = \dfrac{1}{x^2+3} + \dfrac{x}{(x^2+3)^2}}$

Ex.8 continued

$$\int \frac{x^2+x+3}{x^4+6x^2+9}\,dx = \int \left[\frac{1}{x^2+3} + \frac{x}{(x^2+3)^2}\right]dx$$

$$= \int \frac{1}{x^2+3}\,dx + \int \frac{x}{(x^2+3)^2}\,dx$$

$$= \int \frac{1}{x^2+(\sqrt{3})^2}\,dx + \int \frac{x}{(u)^2}\left(\frac{du}{2x}\right)$$

$$= \int \frac{1}{x^2+a^2}\,dx + \frac{1}{2}\cdot\int u^{-2}\,du$$

$$= \frac{1}{a}\arctan\left(\frac{x}{a}\right) + \frac{1}{2}\cdot\left(\frac{u^{-1}}{-1}\right) + C$$

$$= \frac{1}{\sqrt{3}}\arctan\left(\frac{x}{\sqrt{3}}\right) - \frac{1}{2u} + C$$

$$\boxed{= \frac{1}{\sqrt{3}}\arctan\left(\frac{x}{\sqrt{3}}\right) - \frac{1}{2(x^2+3)} + C} \checkmark$$

OR

$$= \frac{\sqrt{3}}{3}\arctan\left(\frac{\sqrt{3}\,x}{3}\right) + \frac{1}{2(x^2+3)} + C$$

Let $u = x^2+3$

$\frac{du}{dx} = 2x$

$\frac{du}{2x} = dx$

Let $a^2 = 3$

$a = \sqrt{3}$

$\int \frac{1}{x^2+a^2}\,dx$

$= \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$

<u>check</u>: ??

Ex.9 Decompose: $\dfrac{3x-5}{x^3-1}$

1st: $x^3-1=(x-1)\underline{(x^2+x+1)}$

↑Irreducible Quadratic Factor

$$\frac{3x-5}{x^3-1}=\frac{3x-5}{(x-1)(x^2+x+1)}=\frac{A}{x-1}+\frac{Bx+C}{x^2+x+1}$$

Degree 0 — Degree 1 — Degree 1 — Degree 2

All Fractions Need to be "Proper"

$$(x-1)(x^2+x+1)\cdot\left[\frac{3x-5}{x^3-1}\right]=(x-1)(x^2+x+1)\cdot\left[\frac{A}{x-1}+\frac{Bx+C}{x^2+x+1}\right]$$

$$\underline{3x-5=A(x^2+x+1)+(x-1)(Bx+C)} \quad \text{(BASIC EQUATION)}$$

$$3x-5=Ax^2+Ax+A+Bx^2-Bx+Cx-C$$

$$3x-5=(Ax^2+Bx^2)+(Ax-Bx+Cx)+(A-C)$$

$$0\cdot x^2+3\cdot x-5=(A+B)\cdot x^2+(A-B+C)\cdot x+(A-C)$$

$\boxed{\text{Match coefficients}}$

$$\begin{cases} A+B+0\cdot C=0 \\ A-B+C=3 \\ A+0\cdot B-C=-5 \end{cases} \longrightarrow \text{MATRIX} \quad \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 3 \\ 1 & 0 & -1 & -5 \end{array}\right]=[F]$$

System of Equations

My Choice for the name

Solve on Ti-83 or Ti-84.

rref([F]) gives us ★★

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{2}{3} \\ 0 & 1 & 0 & \frac{2}{3} \\ 0 & 0 & 1 & \frac{13}{3} \end{array}\right]$$

$A=-\frac{2}{3}$ ←

$B=\frac{2}{3}$ ←

$C=\frac{13}{3}$ ←

use ANS ▶ Frac

to convert matrix entries to fractions

continued

So, we have $\dfrac{3x-5}{x^3-1} = \dfrac{3x-5}{(x-1)(x^2+x+1)}$

$$= \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$$

$$\boxed{\frac{3x-5}{x^3-1} = \frac{-2/3}{x-1} + \frac{\frac{2}{3}x + \frac{13}{3}}{x^2+x+1}}$$

Guidelines for Solving the Basic Equation

Linear Factors

1. Substitute the roots of the distinct linear factors into the basic equation.
2. For repeated linear factors, use the coefficients determined in guideline 1 to rewrite the basic equation. Then substitute other convenient values of x and solve for the remaining coefficients.

Quadratic Factors

1. Expand the basic equation.
2. Collect terms according to powers of x.
3. Equate the coefficients of like powers to obtain a system of linear equations involving A, B, C, and so on.
4. Solve the system of linear equations.